

Expectation Value in Bell's Theorem

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Abstract

We will demonstrate in this paper that Bell's theorem (Bell's inequality) does not really conflict with quantum mechanics, the controversy between them originates from the different definitions for the expectation value using the probability distribution in Bell's inequality and the expectation value in quantum mechanics. We can not use quantum mechanical expectation value measured in experiments to show the violation of Bell's inequality and then further deny the local hidden-variables theory. Considering the difference of their expectation values, a generalized Bell's inequality is presented, which is coincided with the prediction of quantum mechanics.

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Local hidden-variables theory[1] was ever a striking point in the development of quantum mechanics, yet its validity was questioned by Bell. In 1964, Bell proposed a famous inequality[2](or theorem) and asserted that local hidden-variables theory conflicts with quantum mechanics and can not reproduce all the prediction of the latter. This theorem was latterly improved to a new version after Clauser et al.'s experiment[3] of testing local hidden-variables theory, which is suit for the entire family of deterministic and nondeterministic local hidden-variables theory[4]. From then on, Bell's inequality had been discussed widely whatever in theories or experiments. In 1982, Aspect et al.[5] proposed an experiment to test Bell's inequality by use of time-varying analyzers, their experimental results coincide with the prediction of quantum mechanics and indicate Bell's inequality is violated. That is a serious challenge to local hidden-variables theory. However, the debates never stopped[6]. Jaynes ever criticized that the probabilistic reasoning in Bell's theorem does not follow the rules of probability theory[7], Fine discussed the joint distributions and commutativity in Bell theorem[8], there are other viewpoints on Bell's theorem[9, 10], too. Does Bell's theorem really conflict with theory of quantum mechanics? We said: 'no.' The superficial paradox comes from the confusion of definition for the expectation value using the probability distribution in Bell's inequality and the expectation value in quantum mechanics.

Consider two particles A and B which have spin 1/2 and are in quantum mechanical state $|\Psi\rangle_{AB}$. \mathbf{a} and \mathbf{b} are vectors in ordinary three-space, $A(\mathbf{a}, \lambda)$

is the outcome of a measurement on $\sigma_A \cdot \mathbf{a}$ and $B(\mathbf{b}, \lambda)$ is on $\sigma_B \cdot \mathbf{b}$, then the expectation value in Bell's inequality is

$$P(\mathbf{a}, \mathbf{b}) = \int d\lambda \rho(\lambda) A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda), \quad (1)$$

where $\rho(\lambda)$ is the probability distribution in local hidden-variables theory. We can not identify it with the expectation value ${}_{AB}\langle \Psi | (\sigma_A \cdot \mathbf{a})(\sigma_B \cdot \mathbf{b}) | \Psi \rangle_{AB}$ in quantum mechanics as the usual proof on Bell's inequality. In fact, in local hidden-variables theory the latter can be written as

$${}_{AB}\langle \Psi | (\sigma_A \cdot \mathbf{a})(\sigma_B \cdot \mathbf{b}) | \Psi \rangle_{AB} = \int d\lambda \rho(\lambda) {}_{AB}\langle \Psi | (\sigma_A \cdot \mathbf{a})(\sigma_B \cdot \mathbf{b}) | \Psi \rangle_{AB}(\lambda), \quad (2)$$

while the expectation value $P(\mathbf{a}, \mathbf{b})$ in Bell's theorem is

$$P(\mathbf{a}, \mathbf{b}) = \int d\lambda \rho(\lambda) {}_{AB}\langle \Psi | \sigma_A \cdot \mathbf{a} | \Psi \rangle_{AB}(\lambda) {}_{AB}\langle \Psi | \sigma_B \cdot \mathbf{b} | \Psi \rangle_{AB}(\lambda), \quad (3)$$

where we have replaced the outcomes $A(\mathbf{a}, \lambda)$ and $B(\mathbf{b}, \lambda)$ of measurements on particles A and B with ${}_{AB}\langle \Psi | \sigma_A \cdot \mathbf{a} | \Psi \rangle_{AB}(\lambda)$ and ${}_{AB}\langle \Psi | \sigma_B \cdot \mathbf{b} | \Psi \rangle_{AB}(\lambda)$, respectively. The discrepancy between (2) and (3) are obvious, especially to the entangle state. For example, in the case of singlet state, the expression (3) in Bell's theorem is

$$\begin{aligned} P(\mathbf{a}, \mathbf{b}) &= \int d\lambda \rho(\lambda) \frac{1}{2} [A(\uparrow | \sigma_A \cdot \mathbf{a} | \uparrow)_A(\lambda) B(\downarrow | \sigma_B \cdot \mathbf{b} | \downarrow)_B(\lambda) \\ &\quad + A(\downarrow | \sigma_A \cdot \mathbf{a} | \downarrow)_A(\lambda) B(\downarrow | \sigma_B \cdot \mathbf{b} | \downarrow)_B(\lambda) \\ &\quad + A(\uparrow | \sigma_A \cdot \mathbf{a} | \uparrow)_A(\lambda) B(\uparrow | \sigma_B \cdot \mathbf{b} | \uparrow)_B(\lambda) \\ &\quad + A(\downarrow | \sigma_A \cdot \mathbf{a} | \downarrow)_A(\lambda) B(\uparrow | \sigma_B \cdot \mathbf{b} | \uparrow)_B(\lambda)], \end{aligned} \quad (4)$$

and the expectation value (2) in quantum mechanics will become to

$$\begin{aligned} &{}_{AB}\langle \Psi | (\sigma_A \cdot \mathbf{a})(\sigma_B \cdot \mathbf{b}) | \Psi \rangle_{AB} \\ &= \int d\lambda \rho(\lambda) \frac{1}{2} [A(\uparrow | \sigma_A \cdot \mathbf{a} | \uparrow)_A(\lambda) B(\downarrow | \sigma_B \cdot \mathbf{b} | \downarrow)_B(\lambda) \\ &\quad - A(\uparrow | \sigma_A \cdot \mathbf{a} | \downarrow)_A(\lambda) B(\downarrow | \sigma_B \cdot \mathbf{b} | \uparrow)_B(\lambda) \\ &\quad - A(\downarrow | \sigma_A \cdot \mathbf{a} | \uparrow)_A(\lambda) B(\uparrow | \sigma_B \cdot \mathbf{b} | \downarrow)_B(\lambda) \\ &\quad + A(\downarrow | \sigma_A \cdot \mathbf{a} | \downarrow)_A(\lambda) B(\uparrow | \sigma_B \cdot \mathbf{b} | \uparrow)_B(\lambda)], \end{aligned} \quad (5)$$

which clearly show the differences of the expectation values in Bell's theorem and quantum mechanics. These differences originate from the correlation between particles A and B.

As a matter of fact, the expectation value in quantum mechanics contains the correlation between particles A and B, while the expectation value in Bell's theorem is the product of outcomes of measurements on particles A and B, the expectation value $A(\mathbf{a}, \boldsymbol{\lambda})$ of particle A is independent of the expectation value of particle B and conversely, these independent outcomes $A(\mathbf{a}, \boldsymbol{\lambda})$ and $B(\mathbf{b}, \boldsymbol{\lambda})$ of measurements have destroyed the correlation between particles A and B, and there are no correlation in the expectation value $P(\mathbf{a}, \mathbf{b})$ of Bell's theorem, which leads to the differences shown in the above. However, if we chose a non-entangle state $|\Psi\rangle_{AB} = |\uparrow\rangle_A |\downarrow\rangle_B$, the differences between the two expectation values will vanish, because there are no correlation between A and B in the non-entangle state.

It is this confusion between the two expectation values that leads to the inconsistency of Bell's theorem and quantum mechanics. In the usual reasoning of Bell's inequality, the quantity $\int d\lambda \rho(\lambda) [A(\mathbf{a}, \boldsymbol{\lambda})B(\mathbf{b}, \boldsymbol{\lambda}) - A(\mathbf{a}, \boldsymbol{\lambda})B(\mathbf{b}', \boldsymbol{\lambda})]$ is naturally divided as $\int d\lambda \rho(\lambda) \{A(\mathbf{a}, \boldsymbol{\lambda})B(\mathbf{b}, \boldsymbol{\lambda})[1 \pm A(\mathbf{a}', \boldsymbol{\lambda})B(\mathbf{b}', \boldsymbol{\lambda})] - \int d\lambda \rho(\lambda) \{A(\mathbf{a}, \boldsymbol{\lambda})B(\mathbf{b}', \boldsymbol{\lambda})[1 \pm A(\mathbf{a}', \boldsymbol{\lambda})B(\mathbf{b}, \boldsymbol{\lambda})]\}$, where \mathbf{a}' , \mathbf{b}' are the other

vectors in three -space. This rearrangement is reasonable in Bell's theorem because there are no correlation between $A(\mathbf{a}, \boldsymbol{\lambda})$ and $B(\mathbf{b}, \boldsymbol{\lambda})$, while errors will occur in quantum mechanics, in which we can not write out the corresponding expression, i.e.

$$\begin{aligned} & {}_{AB}\langle \Psi | (\sigma_A \cdot \mathbf{a})(\sigma_B \cdot \mathbf{b}) - (\sigma_A \cdot \mathbf{a})(\sigma_B \cdot \mathbf{b}') | \Psi \rangle_{AB} \\ &= {}_{AB}\langle \Psi | (\sigma_A \cdot \mathbf{a})(\sigma_B \cdot \mathbf{b}) | \Psi \rangle_{AB} [1 \pm {}_{AB}\langle \Psi | (\sigma_A \cdot \mathbf{a}')(\sigma_B \cdot \mathbf{b}') | \Psi \rangle_{AB}] \\ & \quad - {}_{AB}\langle \Psi | (\sigma_A \cdot \mathbf{a})(\sigma_B \cdot \mathbf{b}') | \Psi \rangle_{AB} [1 \pm {}_{AB}\langle \Psi | (\sigma_A \cdot \mathbf{a}')(\sigma_B \cdot \mathbf{b}) | \Psi \rangle_{AB}], \end{aligned} \quad (6)$$

which is wrong. We can check it by a simple example. Choosing the angle between vectors \mathbf{a} and \mathbf{b} , \mathbf{b} and \mathbf{c} are 60° in order and \mathbf{a}' and \mathbf{b}' as the same vector \mathbf{c} . According to quantum mechanics, the left side of expression (6) is $-\frac{1}{2} - \frac{1}{2} = -1$, while the right side is $-\frac{1}{2}[1 \pm (-1)] - \frac{1}{2}[1 \pm (-\frac{1}{2})]$, they are not equal. The above rearrangement is violated in quantum mechanics, and the derivation of Bell's inequality can not be resumed considering the correlation in quantum mechanics.

So far, we conclude that Bell's inequality does not really conflict with quantum mechanics, the paradox between them originates from the difference of their expectation values. Bell's inequality is reasonable if we define the expectation value as expression (1) in which the correlation between particles A and B do not exist. However we can not simply compare this expectation value with the one in quantum mechanics because of the above difference. The experimental results did not violated Bell's inequality because it measured the expectation value in quantum mechanics instead of the expectation value in Bell's inequality. We should not deny the local hidden-variables theory by use of this kind of experiments. Local hidden-variables theory should be tested by other ways.

Our analysis have some similarities with Jaynes'[7], Jaynes' main contention was that Bell's factorization for the probability of joint outcomes A and B of the two measurements does not follow from the rules of probability theory.

Bell's factorization is $P(A, B|\mathbf{a}, \mathbf{b}, \mathbf{c}, \boldsymbol{\lambda}) = P(A|\mathbf{a}, \mathbf{c}, \boldsymbol{\lambda})P(B|\mathbf{b}, \mathbf{c}, \boldsymbol{\lambda})$, while the correct factorization should be $P(A, B|\mathbf{a}, \mathbf{b}, \mathbf{c}, \boldsymbol{\lambda}) = P(A|B, \mathbf{a}, \mathbf{b}, \mathbf{c}, \boldsymbol{\lambda})P(B|\mathbf{a}, \mathbf{b}, \mathbf{c}, \boldsymbol{\lambda})$, which is analogous to our analysis. In our treatment, the expectation value containing the correlation between particles A and B can not be divided into the product of two single expectation values for particles A and B, that means ${}_{AB}\langle\Psi|(\sigma_A \cdot \mathbf{a})(\sigma_B \cdot \mathbf{b})|\Psi\rangle_{AB}(\lambda) \neq {}_{AB}\langle\Psi|\sigma_A \cdot \mathbf{a}|\Psi\rangle_{AB}(\lambda){}_{AB}\langle\Psi|\sigma_B \cdot \mathbf{b}|\Psi\rangle_{AB}(\lambda)$. Although the expectation value ${}_{AB}\langle\Psi|(\sigma_A \cdot \mathbf{a})(\sigma_B \cdot \mathbf{b})|\Psi\rangle_{AB}$ here is different from the probability in Jaynes' reasoning and can not be factorized into the product of two terms as Jaynes', there still exist similarities between them.

Due to the above analysis, we can not test Bell's inequality by use of the expectation value containing the correlation between two particles. In Aspect et al.'s experiment, the expectation value ${}_{AB}\langle\Psi|(\sigma_A \cdot \mathbf{a})(\sigma_B \cdot \mathbf{b})|\Psi\rangle_{AB}$ involving the correlation of two photons is measured by the time-varying analyzers, which is not the same quantity $P(\mathbf{a}, \mathbf{b})$ in Bell's inequality, we can not conclude that the experimental results violate Bell's inequality, and this kind of measurements of expectation values can not be used to test Bell's inequality. However, if we measure the expectation values ${}_{AB}\langle\Psi|\sigma_A \cdot \mathbf{a}|\Psi\rangle_{AB}(\lambda)$ and ${}_{AB}\langle\Psi|\sigma_B \cdot \mathbf{b}|\Psi\rangle_{AB}(\lambda)$ for photons A and B separately, there will no correlation between particles A and B be involved in the procedure of measurement, the expectation value $\int d\lambda \rho(\lambda)A(\mathbf{a}, \boldsymbol{\lambda})B(\mathbf{b}, \boldsymbol{\lambda})$ in Bell's inequality can thus be obtained naturally, then we can compare the experimental results with Bell's inequality directly, that will fulfil the test of Bell's inequality. We believe that this kind of experimental results will coincide with Bell's inequality.

Finally, we extend Bell's inequality to a general form in order to match it with the prediction of quantum mechanics. As we know, $\rho(\lambda)$ is the probability distribution of local hidden-variable λ in Bell's theorem, however, if we interpret it as the density matrix of quantum state, which is different from the original definition of $\rho(\lambda)$, we can define the expectation value in Bell's inequality as

$$P(\mathbf{a}, \mathbf{b}) = \text{tr} \left[\int d\lambda \rho(\lambda) (\sigma_A \cdot \mathbf{a})(\sigma_B \cdot \mathbf{b}) \right], \quad (7)$$

which has the similar form with Bell's definition (1), but here it is identical to the quantum mechanics expectation value (2), and the difference of expectation values between Bell's inequality and quantum mechanics will vanish. This generalized expectation value (7) satisfies $-1 \leq \text{tr} \left[\int d\lambda \rho(\lambda) (\sigma_A \cdot \mathbf{a})(\sigma_B \cdot \mathbf{b}) \right] \leq 1$. So we have

$$\begin{aligned} & \left| \text{tr} \left[\int d\lambda \rho(\lambda) (\sigma_A \cdot \mathbf{a})(\sigma_B \cdot \mathbf{b}) \right] - \text{tr} \left[\int d\lambda \rho(\lambda) (\sigma_A \cdot \mathbf{a})(\sigma_B \cdot \mathbf{c}) \right] \right| \quad (8) \\ & + \left| \text{tr} \left[\int d\lambda \rho(\lambda) (\sigma_A \cdot \mathbf{b})(\sigma_B \cdot \mathbf{c}) \right] \right| \\ & \leq 3, \end{aligned}$$

which can be further rearranged as

$$|P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c})| \leq 3 - |P(\mathbf{b}, \mathbf{c})|. \quad (9)$$

So taht we have arrived at our final results- the general Bell's inequality, which has similar form with the original Bell's inequality. Inequality (9) has contained the correlation between particles A and B, and must coincide with the prediction of quantum mechanics.

In summary, we show that Bell's inequality does not conflict with quantum mechanics because of the difference on the expectation values of their own. The experimental results have not violated Bell's inequality. Local hidden variables theory still need to be tested by other ways.

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